DEPARTMENTAL CONFERENCE IN MATHEMATICS[1]

Conducted by
ASSISTANT PROFESSOR H. E. SLAUGHT
The University of Chicago

The number in attendance at the Departmental Conference in Mathematics was about sixty, representing some forty-five schools. The discussion on the topic: "What Should Be the Scope and Purpose of First Year Work in Algebra?" was led by four representatives of widely different relations to the school system, all of whom are engaged in an active campaign for the improvement of teaching algebra and among whom there was substantial agreement on certain fundamental points as follows:

1. There is need for careful reconsideration of the scope and purpose of algebra in the first year of secondary school.

2. At present too much complicated and abstract work in algebra is introduced in the first year. While ample drill work on the processes is absolutely necessary, yet it is more important to drive home the few fundamental principles on oft-repeated simple exercises than utterly to discourage the pupil with continued complicated manipulations which require hours of his study time.

3. Theoretical and demonstrational presentation of algebra is not fitted to the age and maturity of the first-year pupil.

4. The solution of problems should form the central theme for the first year. These should be concrete, interesting, and connecting with things related to life. New processes should be introducd as they are needed in handling problems, and when so introduced should be drilled upon until they are thoroughly familiar. Concrete geometry and physics should furnish large and rich sources for problems. While neither demonstrational geometry nor experimental physics should be taught in the alge-

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bra of the first year, yet the simple and easily recognized facts of both these subjects are easily within the reach of the pupil's comprehension and interest.

5. All of the work of the first year should be developed out of and tied to arithmetic. The chasm between arithmetic and algebra should be broken down.

6. The foregoing points of view compel a readjustment in order of topics. Instead of crowding the first four or five months of the year with abstract and complicated manipulations in long multiplications and divisions, fractions, factoring, highest common divisor, lowest common multiple, etc., for which the pupils never find any use in applications, either here or anywhere, much of this work should be put late in the year (much indeed should be left out entirely from the high-school requirements, and included only in the later course to be elected by those preparing for college).

In this way space and time will be gained for solving problems and for helping both the pupils who may go to college, and those who will not, to see that algebra is not a mere juggling of symbols but is a practical tool for interesting, useful, and immediate service.

Superintendent C. M. Shelton of Crystal Lake, Ill., is working out a plan on these lines both in the high school and in the grades, which he describes briefly as follows:

In a very practical way, we are grappling with two fundamental problems, the isolation of the high school and the disconnected treatment of subjects.

When a pupil reaches the high school, he seems unable to make use of his grade studies in any satisfactory way, and again when he leaves high school to enter college or to go into business, he is confronted with a similar experience. This useless isolation of the high school wastes the time of the teachers and makes the progress of the pupils unsatisfactory. Most rural schools regard their grades as an end in themselves. but city schools cannot look upon the grades and high school as two distinct units unless they blindly follow an outgrown custom. We maintain a continuous system of twelve years, in which the grades reach up into the high school and the high school reaches down into the grades. Similarly, our high school is sutured into the business world and higher schools. The brevity of this paper prohibits any adequate discussion of this last correlation.

In our first six grades, the aim is not to give the pupils a certain proficiency in each subject, but to gain a working control over the tools of learning. We do not *master* a certain set of studies in the first eight years to be dropped for another set during the last four years. By developing the new studies out of the old ones, we keep the child's previous work functional and make his development natural. This gradual growth avoids that abrupt change from familiar subjects to strange ones and obliterates the old break between the grades and the high school.

The disconnected treatment of subjects that we are overcoming, is an old barrier that has grown up through the improvement of separate studies and the development of detached groups. Enthusiasts in all lines improve their subjects without sufficient regard for kindred topics and the time at the disposal of each pupil to study the four groups of subjects. Teachers take these improvements, usually in the form of textbooks, and hitch them to our present school machinery that leads farther than ever from unity and coherence. We are not trying to think in terms of separate textbooks and recitations, nor to regard the one subject that we teach worthy of our notice to the exclusion of all others. We are putting into practical operation a thoroughgoing correlation that attempts to see the entire scheme of unity of each of the four great groups—language, history, science, and mathematics. In order that the reader may see the treatment of one subject, a short account of our work in mathematics is given below.

In the first six years, sufficient arithmetic for the ordinary affairs of life is taught; in the last six years, a mathematical sense and type of thought is developed through arithmetic, geometry, algebra, and trigonometry. There is no attempt to *master* arithmetic in eight years and then to drop it for a succeeding year of algebra which in turn gives way to another in geometry. We recognize no hard-and-fast dividing lines; the differentiation of mathematics into its component parts is left to develop with the child. This work in the higher grades grows out of the pupil's present arithmetical knowledge and develops hand in hand with his advanced arithmetic. In this transition period, arithmetic is not taught one day in the week and algebra and geometry two days each. Such a treatment undoubtedly would mean failure. For instance, when a topic in geometry is taken up, it is not dropped merely to change to algebra or arithmetic. There is a definite relation between the three that determines the length any one topic is developed in its proper setting before it is interpreted into its broader mathematical sense. This seems like a rational treatment of the subject and a natural method for the child. Certainly, clearer ideas are formed and more time is given for drill and problems through the use of our present waste.

The following is a brief of the syllabus which we use:

Grades one and two.—Formal number work is incidental to the development of number idea.

Grades three and four.—Number idea incidental to formal drill in

mechanical processes. The aim in these grades is to develop the habit of using and thinking in terms of signs, symbols, mathematical language, and rapid work in the fundamental operations. Enough applied problems and experimental work is used to give a practical outlook.

Grades five and six.—Problems relating to the everyday affairs of life, reasonable work in fractions, ratio, percentage—profit and loss, commission, and interest, general business applications within the experience or grasp of the child, literal arithmetic and the equation are emphasized. As in other grades, teachers are to note whether a pupil is weak in the abstract and mechanical processes or in the logic and number relations. Pupils are not supposed to get out of these grades weak and lame in either of these points.

Grades seven and eight.—Geometry and algebra developed from and with arithmetic. Geometry, purely experimental and inventional, is used to develop the concepts of forms with their names, to *see* truths without formal proofs. Abundant use is made of exact drawings, paper cuttings and foldings, constructions, superpositions and models with sand.

The algebra begins with literal arithmetic. Addition and substraction are taught together. Concrete interpretation of positive and negative numbers, of the two uses of the plus and minus signs, and then the gradual development of their abstract meanings. Multiplication and division are taught together. Special emphasis is given to the linear equation and to the general algebraic solution of problems by means of the unknown quantity. This is followed by the evaluation of simple functions and straight line graphs; arithmetical fractions and fractional linear equations; special methods in multiplication and factoring developed together.

The work of these two grades is *arithmetical* mathematics. Advanced arithmetic is now begun and subjects formerly presented as arithmetic are taken up in their differentiated setting. Old problems are now re-solved in accordance with the new light. The actual arithmetic is placed in the hands of the pupils who take a delight in reconquering its problems on the basis of algebra. This puts a practical meaning into algebra and makes the pupils firm believers in its merits. It helps the parents to see the value of high-school studies.

Although we do a considerable amount of practical work such as problems from trade, manufacture, and commerce in and about our town, make charts of prices, statistics from our school, and reports of observational groups, we try to keep in mind the value of skill in the manipulation of the abstract and the long apprenticeship in drill that accomplishes it.

Grades nine and ten.—The main axis of the work in the ninth grade is algebra and in the tenth, geometry. The work of both years recognizes the previous work done below. This algebra extends through pure and affected quadratics, and the plane geometry, correlated from time to time

with the solid, is finished. Arithmetic is again kept alive through the special theorems in each.

Grades eleven and twelve.—In the eleventh year, advanced algebra and solid geometry, in which is correlated the beginnings of trigonometry and analytical geometry, occupy the entire year. Those who go into business take a year of arithmetic in the light of their previous mathematical training.

In the twelfth year, physics is presented and the pupil finds an old friend in most of the formulae and equations that he meets. In other words, he is equipped with the tools necessary to progress in physics.

Mr. G. A. Harper at the New Trier Township High School, Wilmette, Ill., is enthusiastic over the problem point of view in algebra as against mere abstract manipulation. He has gathered some data on this point which he presents in brief outline as follows:

Every live student of mathematics is very much interested in the solution of problems. To be able to answer questions and solve practical problems which deal with everyday life is the normal ambition of every healthy-minded boy or girl, man or woman.

Our high schools, by crowding our courses with so many college-entrance requirements, have done much to destroy this ambition. The lists of problems which are to be solved by means of equations are often passed over so hurriedly that the interest in them is entirely lost.

It has always seemed to me that our aim should be to train our pupils to solve these practical problems rather than to put the emphasis upon the abstract principles alone. Too much time is spent in solving equations and not enough is given to the more valuable concrete illustrations. The methods of "checking," which can be made very valuable, are especially "overworked" by the majority of high-school teachers. Very few students of elementary algebra are able to check satisfactorily those problems in which the value of the unknown number is fractional. It would be much better for us to teach our pupils to use the equations in the solution of simple problems, rather than to overemphasize the abstract principles of the equation.

I have written letters to a large number of high-school teachers asking for information concerning their difficulties with these particular problems. One of the questions was, "What percentage of your pupils are able to solve one-half of the written problems without assistance?" Nearly all of the answers ranged from 25 per cent. to 40 per cent. A few were higher than that, and some were lower. One answer was as low as 5 per cent. Does this not show that there is a reason for complaining that these lists are too difficult? When two-thirds of the class are unable to make a reasonable preparation of the lesson, something must be wrong. To be sure we need a few of the difficult problems to maintain the interest of the best students,

but there should be a large number of simpler problems that the majority of the class can solve.

Other questions were, "What types of problems do you consider most practical? Are there any types of problems such as clock problems, digit problems, etc., which you consider impractical?" In answering these questions, nearly all condemned those named as being impractical for the work of the first year, although some acknowledged their usefulness in advanced work. Several expressed the opinion that the problems should deal with practical things, such as the principles of interest, taxes, measurement of areas, and the elementary facts of physics. Several were strong in their condemnation of those problems which have long since lost their interest.

May there not be a reason for the above complaints? Is it not possible that we have difficulty in getting the pupils interested in these problems because we do not talk about interesting things?

But there are many ways of making these problems interesting. The methods that are most practical depend upon the individual teacher. The plan that I have found most successful is to persuade the pupils to write some of these problems for themselves. One will be surprised at the things they will write about. The common everyday occurrences furnish them material for interesting and instructive problems, a large proportion of which will compare favorably with the lists in an ordinary text. Then again, this method is valuable, for it gives the teacher an opportunity to discover the pupils' difficulties with the English language. For these reasons, it would be well to have such work done quite frequently during the first year's work in algebra.

In concluding this discussion, I wish to say that we should emphasize very strongly the practical problems in the work of the first year of algebra. I heartily indorse the report of the Committee of the Central Association on Algebra in the Secondary Schools, but I would go farther and say that we should give the greater part of our time to those principles that are necessary for the solution of all kinds of practical problems. These principles can be made to include nearly everything that is at present required for the first year's work in the average high school.

The problems should include as much mathematical knowledge as the pupil is capable of mastering during the year. They should be simple enough for the majority of the class to solve them readily. Forty easy problems solved by ninety per cent. of the class will do more good than five miserably worded puzzles explained by the teacher.

A year's work conducted on the above plan will produce satisfactory results and will develop an interest in mathematics that will continue throughout the entire high-school course.

Mr. J. H. Dickey at the Academy of James Millikin University, Decatur, Ill., is a strong advocate of reform along the

lines mentioned above and is getting results to justify his theories. His suggestions are given in outline below:

For a long time the plan in elementary algebra has been to teach all the theory possible during the first year, depending upon the pupil to learn for himself how to make the application in the sciences which are presented later. The subject is so comprehensive it has seemed impossible to teach both the theory and its application extensively in the time at our disposal. As a consequence, the theory has received nearly all the attention, while the application to problems has been neglected, very much to the disadvantage of the pupil.

The physics teacher objects to this arrangement. He thinks our pupils too often fail to see any relation between the principles of algebra and the data obtained from a simple experiment in the laboratory; too often they do not know the first steps in transforming a simple statement from words into symbols. He thinks he has to spend too much time teaching elementary algebra when he ought to be teaching physics.

It is a matter of common observation that a treatment 90 per cent. theory and 10 per cent. application, which is found in most textbooks now in use, is one not likely to arouse enthusiasm on the part of the first-year pupil. It is not very strange that he often thinks the subject uninteresting and wonders how it is ever to be of any practical use to him. The argument that a knowledge of the principles involved is necessary for any work in the more advanced mathematics is only partially satisfactory. In the majority of cases he does not expect to continue the study of mathematics very far, so such argument does not make a very strong appeal. There are too many subjects which do make a direct appeal to his life to warrant his spending any great amount of time and energy on a subject which seems to him to be almost wholly theoretical. So, from the point of view of the pupil, it seems there might be a change in the method of presentation.

There is a growing tendency at present to subordinate the theoretical side to the applications of the subject-matter. There is a growing appreciation of the fact that algebra as an abstract science and algebra as a school science are two subjects not at all identical. A logical treatment may be sufficient for the one but not for the other. There must be not only a logical treatment but a psychological treatment as well.

In beginning the study of the violin or the piano the time given to technique the first year must be large in proportion to the amount devoted to the intricacies of musical composition. Instruction in the use of the instrument must be illustrated constantly. The pupil must practice with the instrument in his own hands. A fine theory with no practice would never make a thorough musician. As the pupil develops, harmony and the different forms of musical composition are explained to him. There is an analogy to this in first-year algebra. The development of the theoretical side of the

subject ought to be subordinated at first to the solution of concrete problems involving equations. Those parts of the theory which are attempted should be presented not as being finished products in themselves but simply as helps to the solution of equations. Every principle stated should be accompanied by a set of comparatively easy problems illustrating its application. From beginning to end the stating of equations from written problems should receive the greatest emphasis. This, it seems to me, should dominate the whole of the first year's work. After that a more formal treatment of abstract numbers may be presented to advantage.

The selection of problems is a matter of great importance. Most of them should be concrete, whereas most of them in a majority of textbooks are abstract. They should relate to things with which the pupil is acquainted and should lead very gradually from the concrete to the abstract. We may not expect very much abstract reasoning from a first-year pupil.

In a conversation, a college man who has had twenty years' successful experience teaching mathematics remarked that with each succeeding year he is inclined to demand a trifle less in requirements to enter the Freshman class. He feels that we, as a rule, are inclined to expect a little too much in the way of abstract reasoning from preparatory or high-school pupils.

The question as to where we should turn for material for problems invites discussion. It seems to me our algebras should make more use of the material which might be drawn from manual training. It has been found useful in grammar-school arithmetic and I believe it could be used as well or better in algebra. Statistics from geography, a subject with which the pupil is acquainted, furnish an abundance. Statistics of our industrial resources, simple geometrical drawings, and affairs from everyday life furnish material of high pedagogical value for algebra. Many problems which might seem uninteresting if artificial are interesting because the material in them represents real conditions and has been drawn from real sources. Some types of very elementary problems from physics might be introduced, but problems from physics which require anything more than a very slight explanation should be excluded. If introduced at all, they should be solely for the mathematical content. I do not believe in teaching physics in the algebra class.

Problems should be classified according to subject-matter; that is, problems of the same type should be placed together. A group of problems involving areas, another group involving volumes, etc., followed by miscellaneous lists, give better results than altogether miscellaneous lists. Most of the problems should be of such a nature that no complicated equations result in their solution. Very few complicated equations are necessary to solve any problem in elementary physics. There should be a few problems hard enough, however, to test the best powers of the best members of the class,

otherwise the 15 or 20 per cent. who have naturally the mathematical instinct might think the subject a trifling one.

Formulae are usually best developed from artificial problems. Their use can be learned better from a study of material taken from something with which the pupil is familiar than from data taken from physics. For example, solve for each letter in the formulae for the areas of rectangles, triangles, trapezoids; the volumes of rectangular solids, pyramids; the sides of a right triangle; problems in interest, etc. A large part of the trouble in the use of formulae can be avoided by noting from the beginning the close connection between algebra and arithmetic. Let each new principle be introduced by first studying its application to Arabic figures. Give plenty of exercise involving Arabic figures in the same way as letters. Emphasize daily the fact that the pupil is dealing with actual number expressions and number relations and not merely juggling with letters. For example, adding $5x$ and $10x$ is not identical with the process of adding 5 horses and 10 horses. Make it plain that algebra is generalized arithmetic. Explain all algebraic principles in some reasonable way to pupils, even if they are not able to understand and appreciate the formal abstract proofs.

Checking solutions of equations by substituting particular values for letters, if followed persistently, will accomplish excellent results in the use of formulae. By checking his solution the pupil emphasizes for himself the fact that he has been dealing with actual number expressions and relations in the equation and that the letters are only abbreviations for the numbers. Equations which are too complicated to check easily should not be used much during the first year.

If so much time is devoted to the statement of problems, some topics which are usually presented must now be omitted. For some time I have been omitting from the first year's work H. C. D. and L. C. M. by division, very complicated fractions, simultaneous equations involving more than three unknowns, complicated radicals, and imaginaries. I agree with those who suggest that we might go farther and omit complicated brackets, binomial theorem, cube root of polynomials, simultaneous quadratics except one linear and one quadratic, and theory of quadratics. I should not like to omit the theory of exponents or ratio and proportion. The theory of exponents is not hard to teach and should be taught in connection with radicals.

The introduction of graphs has added an element of interest, for they serve a very useful purpose by showing in a picture the relation between numbers, and give to the analytical solution of equations a very definite meaning. As an illustration of their use, modern texts in preparatory courses in commerce and finance are full of graphs showing curves of exports and imports of products of different kinds. The subject is one of growing importance and I think our first-year pupils should be made acquainted with it so that they may use it in their work as early as possible. It would be going a little

too far, however, to introduce a study of functions in the first year by means of graphs. They should be left until the study of analytics is taken up.

As to the order in which the topics should be presented there seems to be no hard-and-fast rule. Every new text presents the topics in an order different from the last, and yet all are developed in a logical manner. Almost any order found in the average textbook has some peculiar advantages. I would like to commend the plan in one or two of our recent texts of postponing the formal treatment of literal fractions to near the close of the first year, algebraic fractions with numerical denominators being introduced from the first. This arrangement also postpones from its usual place the topic of factoring, which should not be far removed from quadratic equations and fractions.

The plan of dividing the algebra into two separate courses to be given in the first and third years has many good arguments in its favor. The strongest one, it seems to me, is that the maturity of mind necessary for this algebraic work is not generally attained before the third year. In some high schools a term of review in algebra is given in the senior year with good results.

The Decatur High School sends a good many students to the University of Illinois. For some years the Decatur pupils were criticized on account of their poor preparation in algebra. Finally the plan of having a review of algebra in the senior year was adopted. Since then there has been no criticism.

We hope in our school to obtain as good results by another plan. During the first semester of the first year in the academy, arithmetic of somewhat advanced nature is taught. During the second semester a combination course in arithmetic, algebra, and concrete geometry is given. Almost no demonstrational work in geometry is given, but a large amount of constructional and mensurational geometry is introduced. During the second year the algebra is formally finished. During the third year considerable practice is given in the statement and solution of algebraic equations in the plane geometry class, so that by the end of that time the pupil needs no review and is well prepared to begin the work of the Freshman year. We believe this is altogether better than dividing the algebra into two separate courses.

In conclusion, let me suggest again that in our first year, instead of spending so much time on mechanical manipulations with abstract numbers, we proceed at once to the solution of equations, developing the theory as required by them. I believe this will best meet the needs of the pupil, and at the same time furnish the best possible training for him in abstract mathematics.

Supervising Principal Paul G. W. Keller of Manitowoc, Wis., is producing results in the way of added interest and

power in the study of algebra as outlined in the following paper:

That elementary algebra still fails of its highest usefulness because of a lack of a clear understanding of its purpose in the secondary school, is quite clear, and it is the object of this paper to suggest one line of presentation which it is hoped will assist in making algebra what it should be—training in logic, and the development of skill in manipulating this tool and applying its principles.

That first-year algebra should be largely a training in the mastery of fundamental principles, and skill in manipulation and recognition of type algebraic forms, none disputes. That this is the ultimate purpose in this part of the student's mathematical study, however, is too commonly the impression left with him, and often this is the idea held by the teacher, if methods of presenting the subject and exceedingly mechanical results are any criteria. Our plan is to take up briefly the different subjects of algebra dwelling especially upon those portions which have points of contact in geometry. Algebra and geometry are tools which, in the advanced sciences at least, we expect the student to have mastered for the purpose of use aside from the other values of these subjects.

Briefly stated our plan in elementary algebra is as follows:

1. Review the fundamental processes and principles of arithmetic and show how these were used and applied in problems in order that the student may see the relation of processes and principles to their application and their purpose in a field which is familiar to him.

2. Concentrate the attention on the processes and principles by introducing the literal symbol in place of the Arabic symbol, and teach the number system including the negative number. Express simple problem relations in this way. This brings into high relief the processes and principles of arithmetic.

3. Introduce the equation as the algebraic sentence, the tool to be used in all subsequent mathematics to facilitate expression of logical thought. Much of the work during the year will be to gain skill in mastering the processes which are needed to manipulate, not juggle with, this powerful instrument. To aid in fixing the idea of the equation the balance can be used effectively. With the balance, the axioms as applied to positive and negative numbers can be readily illustrated and in this connection it should be clearly borne in mind that we cannot treat negative numbers at all until we have found their equivalents in the simple number system. This thought when brought home to the student will save him plenty of worry over the negative number and its meaning and use.

4. Using these fundamental axioms, the student is to apply them in the solution and simplification of the equation, constantly referring to them as authority for his steps. He will thus acquire the habit of demanding of him-

self rigor in manipulation. As for drill work, instead of using only the usual hit-or-miss forms, we should introduce among others, such forms as will be met later in geometry and physics. That these are formulas the full meaning of which will be learned later, does not in the least detract from their present usefulness for the algebra. In fact, it stimulates curiosity to know how these formulas can be derived from such subjects as geometry, physics, and chemistry.

5. The introduction of more complicated forms in equations leads the pupil to see the need of skill in performing the fundamental operations, using the literal symbols instead of the Arabic figures as in arithmetic. Suitable problems should be chosen to illustrate these new processes as used in equations. These can be gathered from a large field of correlated material, practical arithmetic of business and trades, physics, chemistry, geography, geometry, etc.

6. After the equation has been mastered to this point, a new way of expressing the equation relation is presented, namely the method of the graph, which is soon found to express more completely than does the equation or any other algebraic form, certain important relationships; for example, the graph of the direct and the indirect proportions, as illustrated for instance by Hook's law of the spring balance and Boyle's law of gases. It is well actually to perform these experiments and to formulate the results in the following four ways for the sake, not of the physical relationship, but of the *logical treatment of the facts;* namely, (1) state the facts in words; (2) form an equation developed out of the facts; (3) interpret this equation as a proportion or in the form of a law of variation; (4) graph the law in the form of a picture displaying to the eye the whole range of facts investigated.

The above methods of presentation are so fundamentally important that we can hardly omit them in any treatment of algebra which has as its aim the utility side of the subject.

7. Next in order should come the study of simultaneous equations, and finally general factoring, quadratic equations, and roots. The geometric conceptions aid here greatly in fixing the ideas pertaining to quadratics and roots. Expressing relationships by means of the graph is applicable in many ways where no other method could give an equally complete and comprehensive idea of the conditions. The following are some of the points of contact and application:

In physiology: food-value curve; diet and work curve; localization of skin-sensation curve; alcohol curve of nutrition and intoxication; graphic representation of muscular contraction.

In chemistry: the atomic-weight curve; Boyle's Law curve; Law of Charles curve.

In physical geography: temperature curves for day and month; atmospheric-pressure curves; humidity curves; saturation curves.

In physics: Hook's Law; pressure in free liquid related to height; Boyle's Law; relation between space and time in accelerated motion; the curve of a projectile shot horizontally; vapor-saturation curve; curve representing strength of a magnet as it varies from pole to center, and from center to pole in the opposite direction; law of inverse square in sound or light or force.

In closing let me state briefly a few points which we try to emphasize in our work: (1) that the equation is the core in elementary algebra; that its study is for the purpose of making it a tool in expressing logic simply and rigorously; (2) that there are two other fundamental forms of expressing an equality relation, the proportion and the graph; (3) that in handling equations to solve problems, the student must guard against writing an untruth, by keeping the units in which the elements are expressed consistent; (4) that each step in the manipulation is directly based on the fundamental axioms; (5) that every problem can be handled by one of the following; (*a*) the equation or a set of simultaneous equations, (*b*) the simple proportion, (*c*) the compound proportion; (6) that the secret of solving a problem lies (*a*) in finding what elements are wanted, (*b*) in discovering the relations of the given elements to those wanted, (*c*) in knowing that there must be found as many relationships involving the known and unknown elements as there are elements wanted.